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Abstract
Using a reformulation of the method of (p, q) webs, we study the four-dimensional $N = 1$ quiver theories from M-theory on seven-dimensional manifolds with G_2 holonomy. We first construct such manifolds as $U(1)$ quotients of eight-dimensional toric hyper-Kähler manifolds, using $N = 4$ supersymmetric sigma models. We show that these geometries, in general, are given by real cones on $\mathbf{S}^2$ bundles over complex two-dimensional toric varieties, $\mathbf{V}^2 = \mathbf{C}^{r+2}/\mathbf{C}^{*r}$. Then we discuss the connection between the physics content of M-theory on such G_2 manifolds and the method of (p, q) webs. Motivated by a result of Acharya and Witten (*Preprint* hep-th/0109152), we reformulate the method of (p, q) webs and reconsider the derivation of the gauge theories using toric geometry Mori vectors of $\mathbf{V}^2$ and brane charge constraints. For $\mathbf{WP}^2_{w_1,w_2,w_3}$, we find that the gauge group is given by $G = U(w_1 n) \times U(w_2 n) \times U(w_3 n)$. This is required by the anomaly cancellation condition.

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1. Introduction

Since the discovery of superstring dualities, four-dimensional supersymmetric quantum field theories (QFT_4) have been a subject of great interest in connection with superstring compactification on Calabi–Yau manifolds and D-brane physics [1–4]. For example, embedding $N = 2QFT_4$ in type IIA superstring compactified on Calabi–Yau threefolds, with K3 fibration, has found a very nice geometric description using the so-called geometric engineering method [5–10]. In this programme, these models, which give exact results for the moduli space of the type IIA Coulomb branch, are represented by Dynkin quiver diagrams of Lie algebras [7–10].

Quite recently, a special interest has been devoted to four-dimensional gauge models preserving only four supercharges [11, 12]. These field models admit a very nice description in the so-called (p, q) webs [13–24]. This method concerns the study of $N = 1$ four-dimensional quiver theories arising on the worldvolume of D3-branes transverse to singular Calabi–Yau threefolds, CY_B^3. The subscript here refers to type IIB string geometry. The manifolds are complex cones over complex two-dimensional toric varieties $\mathbf{V}^2$, e.g. del Pezzo surfaces. They are mirror manifolds of local Calabi–Yau threefolds CY_A^3 given by elliptic and C^* fibrations over the complex plane. Under local mirror symmetry, a D3-brane in type IIB geometry becomes a D6-brane wrapping a T^3 in type IIA mirror geometry. In this way, the $N = 1$ four-dimensional quiver theories can be obtained from D6-branes wrapping 3-cycles S_i in the mirror manifold. For instance, a D6-brane on T^3, whose homology class is

$$[T^3] = \sum_{i=1}^{\ell} n_i S_i \tag{1.1}$$

where $\{S_i, i = 1, \ldots, \ell\}$ form a basis of $H_3(CY_A^3, Z)$, gives a four-dimensional $N = 1$ supersymmetric gauge theory with a gauge group

$$G = \prod_{i=1}^{\ell} U(n_i) \tag{1.2}$$

and quiver matrix

$$I_{ij} = S_i \cdot S_j. \tag{1.3}$$

In equations (1.1) and (1.2), the vector n_i is specified by the anomaly cancellation condition

$$\sum_{i=1}^{\ell} I_{ij} n_i = 0. \tag{1.4}$$

The above identities in the method of (p, q) webs are very exciting. First, the same equation forms have been used in the geometric engineering of superconformal models with eight supercharges. In this case, the quiver matrix is identified with an affine *ADE* Cartan matrix K and the gauge group is $G = \prod_i SU(s_i n)$. The positive integers s_i appearing in G are the usual Dynkin weights. They form a special positive definite integer vector $s = (s_i)$ satisfying $K_{ij} s_j = 0$, as required by the vanishing of the beta function. Second, for $\ell = 3$ corresponding to complex two-dimensional weighted projective spaces in type IIB geometry, the physics content with unitary gauge groups and charged chiral matter seems to be similar to four-dimensional $N = 1$ models obtained from M-theory on singular G_2 manifolds studied first in [25], see also [26, 27]. These manifolds are constructed as circle quotients of eight-dimensional toric hyper-Kähler (HK) manifolds. Following [25], the twistor space over the weighted projective space $\mathbf{WP}^2_{m,m,n}$ has an interpretation in type IIA superstring as an intersection of three groups of D6-branes with multiplicities m, m, n leading to $SU(m) \times SU(m) \times SU(n)$ gauge symmetry. According to this feature, one might ask the following question. Is there a connection between the approach of (p, q) webs and M-theory on G_2 manifolds[1]? However, this connection may naturally lead to the need of a reformulation of the method of (p, q) webs. The reason for this is that the gauge symmetry in the M-theory compactification involves the weights of the weighted projective space $\mathbf{WP}^2$. In this paper, we address this question using toric geometry data of G_2 manifolds as $U(1)$ quotients of eight-dimensional HK manifolds,

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[1] Besides this similarity, much of the D6-branes physics content can be interpreted in M-theory on local G_2 manifolds, leading to $N = 1$ supersymmetric models in four dimensions.

and by reconsidering the method of (p, q) webs. This study may complete the analysis of [28] dealing with discrete G_2 orbifolds using the McKay correspondence [29].

Our programme will proceed in two steps:

(i) We study G_2 manifolds as $U(1)$ quotients of eight-dimensional toric HK manifolds, $X_7 = X_8/U(1)$. The manifold X_8 is obtained using relevant constraint equations in terms of two-dimensional $N = 4$ sigma models with $U(1)^r$ gauge symmetry and $r + 2$ hypermultiplets [25, 26, 30]. We show that the resulting seven-dimensional manifolds, in general, are given by real cones on $\mathbf{S}^2$ bundles over complex two-dimensional toric varieties

$$\mathbf{V}^2 = \mathbf{C}^{r+2}/\mathbf{C}^{*r}. \tag{1.5}$$

Explicit models are presented in terms of two-dimensional $N = 2$ sigma model realizations of $\mathbf{V}^2$.

(ii) We discuss the link between the physics content of M-theory on such G_2 manifolds and the methods of (p, q) webs. In particular, we reconsider and reformulate the (p, q) web equations using the toric geometry Mori vectors of $\mathbf{V}^2$ and set of brane charge constraint equations. For the weighted projective space $\mathbf{WP}^2_{w_1, w_2, w_3}$, for example, we find the following gauge group:

$$G = U(w_1 n) \times U(w_2 n) \times U(w_3 n). \tag{1.6}$$

This is required by the anomaly cancellation condition. With an appropriate choice of weight vectors, we recover the result of Acharya and Witten given in [25].

The plan of this paper is as follows. In section 2, we briefly review the main lines of the toric geometry method for treating complex manifolds. Then we give the interplay between the toric geometry and two-dimensional $N = 2$ supersymmetric gauge theories. In section 3, we study G_2 manifolds as $U(1)$ quotients of eight-dimensional toric HK manifolds X_8 constructed from D-flatness conditions of two-dimensional field theory with $N = 4$ supersymmetry. Then we identify the $U(1)$ symmetry group with the toric geometry circle actions of X_8 to present quotients $X_7 = X_8/U(1)$ of G_2 holonomy. Explicit models are given in terms of real cones on an $\mathbf{S}^2$ bundle over complex two-dimensional toric varieties $\mathbf{V}^2$. In section 4, we engineer $N = 1$ quiver models from G_2 manifolds. We discuss the link between the physics content of M-theory on such G_2 manifolds and the method of (p, q) webs. We reconsider and reformulate the (p, q) equations using the toric geometry Mori vectors of $\mathbf{V}^2$ and set of brane charge constraint equations. In particular, for the weighted projective space $\mathbf{WP}^2_{w_1, w_2, w_3}$, we find that the gauge group is given by (1.6). In section 5, we give illustrating applications. In section 6, we give our conclusion.

2. Toric geometry

In this section, we collect a few facts on toric geometry of complex manifolds. These facts are needed later to construct a special type of G_2 manifold, as $U(1)$ quotients of eight-dimensional toric HK manifolds. Roughly speaking, toric manifolds are complex n-dimensional manifolds with T^n fibration over n-dimensional base spaces with boundary [7, 10, 31–34]. They exhibit toric actions $U(1)^n$ allowing us to encode the geometric properties of the complex spaces in terms of simple combinatorial data of polytopes Δ_n of the R^n space. In this correspondence, fixed points of the toric actions $U(1)^n$ are associated with the vertices of the polytope Δ_n, the edges are fixed one-dimensional lines of a subgroup $U(1)^{n-1}$ of the toric action $U(1)^n$, and so on. Geometrically, this means that the T^n fibres can degenerate over the boundary of the

base. Note that in the case where the base space is compact, the resulting toric manifold will be compact as well.

In string theory, the power of the toric geometry representation is due to the following points. (1) The toric data of the polytope Δ_n have similar features to the *ADE* Dynkin diagrams leading to non-Abelian gauge symmetries in type II superstring compactifications on Calabi–Yau manifolds [7–10]. (2) The toric fixed loci, which correspond to the vanishing cycles, have been known to be associated with D-brane charges [32]. The latter will be used in section 4 to discuss the physics content of M-theory on our proposed manifolds of G_2 holonomy, using a reformulation of the method of (p, q) webs in type II superstring on Calabi–Yau threefolds.

To illustrate the main idea of toric geometry, let us describe the philosophy of this subject through certain useful examples.

(i) $\mathbf{P}^1$ *projective space*.

This is the simplest example in toric geometry which turns out to play a crucial role in the building blocks of higher-dimensional toric varieties and in the study of the small resolution of *ADE* singularities of local Calabi–Yau manifolds. $\mathbf{P}^1$ has a $U(1)$ toric action

$$z \to e^{i\theta} z \tag{2.1}$$

with two fixed points v_1 and v_2 on the real line. The latter points, which can be generally chosen as $v_1 = -1$ and $v_2 = 1$, describe respectively north and south poles of the real two sphere $\mathbf{S}^2 \sim \mathbf{P}^1$. The corresponding one-dimensional polytope is just the segment $[v_1, v_2]$ joining the two points v_1 and v_2. Thus, $\mathbf{P}^1$ can be viewed as a segment $[v_1, v_2]$ with a circle on top, where the circle vanishes at the endpoints v_1 and v_2.

(ii) $\mathbf{P}^2$ *projective space*.

$\mathbf{P}^2$ is a complex two-dimensional toric variety defined by

$$\mathbf{P}^2 = \frac{C^3 \backslash \{(0, 0, 0)\}}{C^*} \tag{2.2}$$

where C^* acts as follows:

$$(z_1, z_2, z_3) \to (\lambda z_1, \lambda z_2, \lambda z_3). \tag{2.3}$$

It admits a $U(1)^2$ toric action

$$(z_1, z_2, z_3) \to (e^{i\theta_1} z_1, e^{i\theta_2} z_2, z_3) \tag{2.4}$$

exhibiting three fixed points v_1, v_2 and v_3. The corresponding polytope Δ_2 is a finite sublattice of the $\mathbf{Z}^2$ square lattice. It describes the intersection of three $\mathbf{P}^1$ defining a triangle $(v_1 v_2 v_3)$ in the $\mathbf{R}^2$ plane. A convenient choice of the data of these three vertices is as follows : $v_1 = (1, 0)$, $v_2 = (0, 1)$ and $v_3 = (-1, -1)$. Thus, Δ_2 has three edges, namely $[v_1, v_2]$, $[v_2, v_3]$ and $[v_3, v_1]$ stable under the three $U(1)$ subgroups of $U(1)^2$; two subgroups are just the two $U(1)$ factors, while the third subgroup is the diagonal one. $\mathbf{P}^2$ can be viewed as a triangle over each point of which there is an elliptic curve T^2. This torus shrinks to a circle at each segment $[v_i, v_j]$ and it shrinks to a point at each v_i. The above toric realization can be pushed further for describing the same phenomenon involving complex n-dimensional toric varieties that are more complicated than projective spaces. The latter spaces can be expressed in the following form:

$$\mathbf{V}^n = \frac{C^{n+r} \backslash U}{C^{*r}} \tag{2.5}$$

where now we have $r C^*$ actions given by

$$C^{*r} : z_i \to \lambda^{Q_i^a} z_i \qquad i = 1, 2, \ldots, n + r \qquad a = 1, 2, \ldots, r. \tag{2.6}$$

In this equation, Q_i^a are integers. For each a they form the so-called Mori vectors in toric geometry. They generalize the weight vector (w_i) of the complex n-dimensional weighted projective space $\mathbf{WP}^n_{w_1,\ldots,w_{n+1}}$. U is a subset of C^k chosen by triangulation [7].

Equation (2.5) means that $\mathbf{V}^n$ has a T^n fibration, obtained by dividing T^{n+r} by the $U(1)^r$ gauge symmetry

$$z_i \to e^{iQ_i^a \vartheta^a} z_i \qquad a = 1, \ldots, r \tag{2.7}$$

where ϑ^a are the generators of the $U(1)$ factors. $\mathbf{V}^n$ can be represented by a toric diagram $\Delta(\mathbf{V}^n)$ spanned by $k = n + r$ vertices v_i of a $\mathbf{Z}^n$ lattice satisfying

$$\sum_{i=1}^{n+r} Q_i^a v_i = 0 \qquad a = 1, \ldots, r. \tag{2.8}$$

The toric geometry manifolds we have been describing have an interesting realization through linear sigma models, where one considers two-dimensional supersymmetric $N = 2$ gauge systems with $U(1)^r$ gauge group and $n + r$ chiral fields X_i with a Q_i^a matrix charge [35]. In this way, the Kähler manifold $\mathbf{V}^n$ is the minimum of the D-term potential ($D^a = 0$), up to $U(1)^r$ gauge transformations, namely

$$\sum_{i=1}^{n+r} Q_i^a |X_i|^2 = R_a \tag{2.9}$$

where the R_a are Fayet–Iliopoulos (FI) coupling parameters. The (local) Calabi–Yau condition is satisfied by

$$\sum_{i=1}^{n+r} Q_i^a = 0 \qquad \forall a \tag{2.10}$$

which means that the system flows in the infrared to a non-trivial superconformal theory [35, 36]. Under local mirror symmetry, this toric Calabi–Yau sigma model maps to Landau–Ginsburg (LG) models [37–40]. In this way, the mirror version of the constraint equation (2.9), giving the LG superpotential, reads

$$\sum_i y_i = 0 \tag{2.11}$$

subject to

$$\prod_i y_i^{Q_i^a} = e^{-t_a} \tag{2.12}$$

where y_i are LG dual chiral fields which can be related, up to some field changes, to sigma model fields, and where t_a are the complexified FI parameters defining now the complex deformations of the LG Calabi–Yau superpotentials.

Note that the above two-dimensional $N = 2$ toric sigma models can be extended to $N = 4$ supersymmetry models with hypermultiplets leading to toric HK geometries [41]. In the rest of this paper, we will use toric geometry and HK analysis to study seven-dimensional manifolds with G_2 holonomy. The latter are $U(1)$ quotients of eight-dimensional toric HK manifolds X_8.

3. G_2 manifolds as $U(1)$ quotients

3.1. G_2 manifolds and $N = 4$ D-flatness conditions

It is known that in order to get a semi-realistic four-dimensional theory from M-theory it is necessary to consider a compactification on a seven-dimensional manifold X_7 with G_2

holonomy [42–51]. In this way, the resulting models with $N = 1$ supersymmetry depend on the geometric properties of X_7. For instance, if X_7 is smooth, the low-energy theory contains, in addition to $N = 1$ supergravity, only Abelian gauge symmetry and no charged chiral fermions. Non-Abelain gauge symmetries can be obtained by considering limits where X_7 develops *ADE* orbifold singularities using wrapped M2-branes on vanishing 2-cycles [43]. However, the presence of conical singularities leads to charged chiral fermions. Following [25], an interesting analysis for building such geometries is to consider quotients of eight-dimensional toric HK manifolds X_8 by a $U(1)$ circle symmetry. The $U(1)$ group has been chosen such that it commutes with the $SU(2)$ symmetry, permuting the three complex structures of HK geometries. *A priori*, there are many ways to choose the $U(1)$ group action. Two situations have been given in [25] but here we will identify the $U(1)$ group with the toric geometry circle action of complex subvarieties within HK geometries. In particular, we will use the HK analysis to present explicit models with G_2 holonomy group leading to interesting $N = 1$ supersymmetric gauge theories in four dimensions. To do so, we consider two-dimensional $N = 4$ supersymmetric gauge theories with $U(1)^r$ gauge symmetries and $r+2$ hypermultiplets with a Q_i^a matrix charge [36, 41]. The $N = 4D$-flatness equations of such models are generally given by

$$\sum_{i=1}^{r+2} Q_i^a \big[\phi_i^\alpha \bar{\phi}_{i\beta} + \phi_{i\beta} \bar{\phi}_i^\alpha\big] = \vec{\xi}_a \vec{\sigma}_\beta^\alpha \qquad a = 1, \ldots, r. \tag{3.1}$$

In these equations, ϕ_i^α denote $r + 2$ component field doublets of hypermultiplets, $\vec{\xi}_a$ are r FI 3-vector couplings rotated by the $SU(2)$ symmetry, and $\vec{\sigma}_\beta^\alpha$ are the traceless 2×2 Pauli matrices. In this construction, for each $U(1)$ factor, there are three real constraint equations transforming as an iso-triplet of the $SU(2)R$-symmetry $(SU(2)_R)$ acting on the HK structures.

Using the $SU(2)_R$ transformations

$$\phi^\alpha = \varepsilon^{\alpha\beta}\phi_\beta \qquad \overline{(\phi^\alpha)} = \overline{\phi}_\alpha \qquad \varepsilon_{12} = \varepsilon^{21} = 1 \tag{3.2}$$

and replacing the Pauli matrices by their expressions, the identities (3.1) can be split as follows:

$$\sum_{i=1}^{k} Q_i^a \big(|\phi_i^1|^2 - |\phi_i^2|^2\big) = \xi_a^3 \tag{3.3}$$

$$\sum_{i=1}^{k} Q_i^a \phi_i^1 \overline{\phi}_i^2 = \xi_a^1 + \mathrm{i}\xi_a^2 \tag{3.4}$$

$$\sum_{i=1}^{k} Q_i^a \phi_i^2 \overline{\phi}_i^1 = \xi_a^1 - \mathrm{i}\xi_a^2. \tag{3.5}$$

Dividing the resulting space of (3.3)–(3.5) by $U(1)^r$ gauge transformations, we find precisely an eight-dimensional toric HK manifold X_8. However, explicit solutions of these geometries depend on the values of the FI couplings. Taking $\xi_a^1 = \xi_a^2 = 0$ and $\xi_a^3 > 0$, (3.3)–(3.5) describe the cotangent bundle over complex two-dimensional toric varieties [30]. Indeed, if we set all $\phi_i^2 = 0$, we get a complex two-dimensional toric variety $\mathbf{V}^2$ defined by $\sum_{i=1}^{2+r} Q_i^a |\phi_i^1|^2 = \xi_a^3, (a = 1, \ldots, r)$. Equations (3.4)–(3.5) mean that the ϕ_i^2 define the cotangent orthogonal fibre directions over $\mathbf{V}^2$. This manifold has four toric geometry circle actions: $U(1)_{\text{base}}^2 \times U(1)_{\text{fibre}}^2$. Two of them correspond to the $\mathbf{V}^2$ toric base space denoted by $U(1)_{\text{base}}^2$, while the remaining ones, $U(1)_{\text{fibre}}^2$, act on the fibre orthogonal cotangent directions. To get the corresponding seven-dimensional manifolds with G_2 holonomy, we will identify the $U(1)$ group symmetry of the quotient used in [25] with one finite circle toric action.

Identifying this $U(1)$ symmetry with one $U(1)_{\text{fibre}}$, one finds the following seven-dimensional manifold

$$X_7 = X_8/U(1)_{\text{fibre}}. \tag{3.6}$$

Since $\mathbf{C}^2/U(1) = \mathbf{R} \times \mathbf{S}^2$, this quotient space is now isomorphic to an $\mathbf{R} \times \mathbf{S}^2$ bundle over a $\mathbf{V}^2$. Similar to [25], equation (3.6) describes real cones on an $\mathbf{S}^2$ bundle over $\mathbf{V}^2$. Mathematically, it is not easy to reveal that these quotient spaces have G_2 holonomy group. However, one can show this using a physical argument. Indeed, $\mathbf{V}^2$, with $h^{1,0} = h^{2,0} = 0$, preserves 1/4 of initial supercharges and in the presence of $\mathbf{S}^2$ it should be 1/8. In this way, the supersymmetry tells us that the holonomy of (3.6) is the G_2 Lie group. Thus, M-theory on the above seven-dimensional manifold leads to $N = 1$ theory in four dimensions.

3.2. Explicit models from $\mathbf{V}^2$ geometries

To better understand the structure of (3.3)–(3.6), let us give illustrating models. In particular, we will consider special models corresponding to $N = 4$ sigma model with conformal invariance. For this reason, we will restrict ourselves to eight-dimensional toric HK manifolds X_8 with the Calabi–Yau condition (2.10) in $N = 4$ supersymmetric analysis. In this way, the geometry of X_8 depends on the manner we choose the $U(1)^r$ matrix gauge charge Q_i^a satisfying the Calabi–Yau condition. We first study complex two-dimensional weighted projective spaces $\mathbf{WP}^2$, after which we will consider the Hirzebruch surfaces. Other extended models are also presented.

3.2.1. $\mathbf{V}^2$ as weighted projective spaces.

For constructing these models, we consider a $U(1)$ gauge symmetry with three hypermultiplets ϕ_i of charges (Q_1, Q_2, Q_3) such that $Q_1 + Q_2 + Q_3 = 0$. One way to solve this constraint equation is to take $Q_1 = m_1$, $Q_2 = -m_1 - m_2$ and $Q_3 = m_2$. This gives $\mathbf{WP}^2_{m_1,m_1+m_2,m_2}$ as a base geometry in the G_2 manifold. Using examples, let us see how we obtain this geometry.

Example 1. $(m_1, m_2) = (1, 1)$. This example corresponds to three hypermultiplets ϕ_i with the vector charge $Q_i = (1, -2, 1)$. After permuting the role of ϕ_2^1 and $\overline{\phi^2}_2$ and making the following field changes $\phi_1^1 = \varphi_1, \phi_1^2 = \psi_1, \phi_3^1 = \varphi_2, \phi_3^2 = \psi_3, -\overline{\phi}_2^2 = \varphi_2, \overline{\phi}_2^1 = \psi_2$, equations (3.3)–(3.5) become

$$(|\varphi_1|^2 + |\varphi_3|^2 + 2|\varphi_2|^2) - (|\psi_1|^2 + |\psi_3|^2 + 2|\psi_2|^2) = \xi^3 \tag{3.7}$$

$$\varphi_1\overline{\psi_1} + \varphi_3\overline{\psi_3} + 2\varphi_2\overline{\psi_2} = 0 \tag{3.8}$$

$$\overline{\varphi_1}\psi_1 + \overline{\varphi_3}\psi_3 + 2\overline{\varphi_2}\psi_2 = 0. \tag{3.9}$$

These equations describe a cotangent bundle over $\mathbf{WP}^2_{1,2,1}$. Indeed, taking $\psi_1 = \psi_2 = \psi_3 = 0$, equation (3.7) reduces to $|\varphi_1|^2 + |\varphi_3|^2 + 2|\varphi_2|^2 = \xi^3$ and defines a $\mathbf{WP}^2_{1,2,1}$ weighted projective space, where ξ^3 is a Kähler real parameter controlling its size. For generic values of ψ_i, equations (3.7)–(3.9) can be interpreted to mean that ψ_i parametrize the orthogonal fibre directions on $\mathbf{WP}^2_{1,2,1}$. Dividing by one finite toric geometry fibre circle action, we find a real cone on an $\mathbf{S}^2$ bundle over $\mathbf{WP}^2_{1,2,1}$ with G_2 holonomy.

Example 2. $(m_1, m_2) = (1, 2)$. As another example, we consider a vector charge as follows $Q_i = (1, -3, 2)$. This example is quite similar to the first one, and its treatment will be parallel to the first one. After making similar field changes, this example describes $\mathbf{WP}^2_{1,3,2}$ in the base geometry of an eight-dimensional manifold. After the $U(1)$ quotient, the corresponding

seven-dimensional manifold X_7 will be a real cone on the $\mathbf{S}^2$ bundle over $\mathbf{WP}^2_{1,3,2}$. We will see later that this geometry leads to a four-dimensional model which might be related to the grand unified symmetry.

3.2.2. $\mathbf{V}^2$ as Hirzebruch surfaces $\mathbf{F}_n$. $\mathbf{F}_n$ are complex two-dimensional toric surfaces defined by non-trivial fibrations of a $\mathbf{P}^1$ over a $\mathbf{P}^1$. These may be viewed as the compactification of complex line bundles over $\mathbf{P}^1$ by adding a point to each fibre at infinity. Such line bundles are classified by an integer n, the first Chern class being integrated over $\mathbf{P}^1$. For simplicity, we will restrict ourselves to $\mathbf{F}_0$ with a trivial fibration. A way to write down the $\mathbf{F}_0$ $N = 4$ sigma model is to start with one $\mathbf{P}^1$ and then extend the result to $\mathbf{F}_0$. Indeed, one $\mathbf{P}^1$ corresponds to a $U(1)$ two-dimensional $N = 4$ linear sigma model with two hypermultiplets with a vector charge $(1, -1)$. Making a similar analysis of previous examples, the D-flatness conditions (3.1) reduce to

$$(|\varphi_1|^2 + |\varphi_2|^2) - (|\psi_1|^2 + |\psi_2|^2) = \xi^3 \tag{3.10}$$

$$\varphi_1\overline{\psi}_1 + \varphi_2\overline{\psi}_2 = 0 \tag{3.11}$$

$$\psi_1\overline{\varphi_1} + \psi_2\overline{\varphi_2} = 0 \tag{3.12}$$

and describe the cotangent bundle over a $\mathbf{P}^1$, defined by $|\varphi_1|^2 + |\varphi_2|^2 = \xi^3$. The model corresponding to $\mathbf{F}_0$ is obtained by considering a $U(1)^2$ two-dimensional $N = 4$ linear sigma model with four hypermultiplets with the following charges:

$$Q_i^{(1)} = (1, -1, 0, 0) \qquad Q_i^{(2)} = (0, 0, 1, -1). \tag{3.13}$$

In this way, $N = 4$ D-flatness constraint equations describe the cotangent bundle over $\mathbf{F}_0$. After dividing by one finite toric geometry circle action, we get a real cone on the $\mathbf{S}^2$ bundle over $\mathbf{F}_0$.

3.3. Other models from $\mathbf{WP}^2$

Here, we study some extended models using more general $N = 4$ two-dimensional gauge theories. In particular, we consider two possible generalizations for $\mathbf{WP}^2$. The first model describes the blowing up of $\mathbf{WP}^2$ at one point. It has a similar feature to the $\mathbf{F}_2$ geometry. The second model deals with a model with *ADE* Cartan matrix gauge charges leading to *ADE* intersecting geometries.

3.3.1. Blowing up of $\mathbf{WP}^2$ at one point. For simplicity, we consider $\mathbf{WP}^2_{1,2,1}$ as an example. This space has a Z_2 orbifold singularity corresponding to non-trivial fixed points under the homogeneous identification

$$(z_1, z_2, z_3) \equiv (\lambda z_1, \lambda^2 z_2, \lambda z_3). \tag{3.14}$$

Taking $\lambda = -1$, $\mathbf{WP}^2_{1,2,1}$ has a Z_2 orbifold singularity at $(z_1, z_2, z_3) = (0, 1, 0)$. This singularity may be blown up by introducing an exceptional divisor. In the two-dimensional $N = 2$ sigma model, this can be deformed by introducing an extra chiral field X_4 and a $U(1)$ gauge group factor. In this way, the corresponding eight-dimensional manifolds can be described by a $U(1)^2$ linear sigma model with four hypermultiplets with the following charges:

$$Q_i^{(1)} = (1, -2, 1, 0) \qquad Q_i^{(2)} = (0, -1, 0, 1) \tag{3.15}$$

This model gives the same G_2 manifold corresponding to the $\mathbf{F}_2$ Hirzebruch surface.

3.3.2. ADE intersecting geometry. Another generalization is to consider the intersecting weighted projective spaces according to *ADE* Dynkin diagrams by imitating the analysis of the $N = 2$ sigma model. This involves two-dimensional $N = 4$ supersymmetric $U(1)^r$ gauge theory with $(r + 2)$ ϕ_i^α hypermultiplets with *ADE* Cartan matrices as matrix gauge charges. For simplicity, let us consider the A_r Lie algebra where the matrix charge is given by $Q_i^a = -2\delta_i^a + \delta_{i-1}^a + \delta_{i+1}^a, a = 1, \ldots, r$. Putting these equations into the *D*-flatness equations (3.1), one gets the following system of $3r$ equations:

$$\left(\left|\phi_{a-1}^1\right|^2 + \left|\phi_{a+1}^1\right|^2 - 2\left|\phi_a^1\right|^2\right) - \left(\left|\phi_{a-1}^2\right|^2 + \left|\phi_{a+1}^2\right|^2 - 2\left|\phi_a^2\right|^2\right) = \xi_a \qquad (3.16)$$

$$\phi_{a-1}^1\overline{\phi_{a-1}^2} + \phi_{a+1}^1\overline{\phi_{a+1}^2} - 2\phi_a^1\overline{\phi_a^2} = 0 \qquad (3.17)$$

$$\phi_{a-1}^2\overline{\phi_{a-1}^1} + \phi_{a+1}^2\overline{\phi_{a+1}^1} - 2\phi_a^2\overline{\phi_a^1} = 0. \qquad (3.18)$$

An examination of these equations reveals that $\mathbf{V}^2$ consists of r intersecting $\mathbf{WP}^2_{1,2,1}$ according to the A_r Dynkin diagram [30]. Actually, this geometry generalizes the usual *ADE* geometry corresponding to 2-cycles of K3 surfaces [7–10]. One expects to have a similar feature in the compactification of M-theory on G_2 manifolds with intersecting $\mathbf{WP}^2_{1,2,1}$.

The previous analysis is also possible for models with del Pezzo surfaces as a base geometry of G_2 manifolds. Note that, these surfaces have been used in the building of $N = 1$ supersymmetric gauge theories in four dimensions using the so-called (p, q) webs. These gauge theories arise on the worldvolume of D3-branes transverse to local Calabi–Yau threefolds CY_B^3 given by complex cones over del Pezzo surfaces [14, 22]. In this work, we will show that this physics is related to M-theory on G_2 manifolds with two complex dimension toric manifolds in the base geometry.

4. On M-theory on G_2 manifolds and (p, q) webs

So far, we have constructed a special type of G_2 manifold as $U(1)$ quotients of eight-dimensional toric HK manifolds. This section will be concerned with M-theory on such manifolds. We will try to find a superstring interpretation of this using D-brane physics. In particular, we will discuss the connection between the physics content of M-theory on such G_2 manifolds and the method of (p, q) webs, leading to $N = 1$ supersymmetric gauge theories in four dimensions. The analysis we use here is based on a reconsideration of the method of (p, q) webs and a reformulation of the intersection number structures in terms of the toric geometry data of $\mathbf{V}^2$ varieties.

Before proceeding, let us recall and quote some crucial points supporting our discussion. On one hand, according to [25], M-theory on G_2 manifolds as the $U(1)$ quotient of eight-dimensional conical toric HK manifolds, has an interpretation in terms of intersecting D6-branes in a type IIA superstring model. In particular, the geometry of $\mathbf{WP}^2$ describes the intersection of three sets of D6-branes. For example, the geometry $\mathbf{WP}^2_{m,m,n}$ with m, m and n relatively prime corresponds to a pair of two spheres of A_m singularities and a single of A_n singularities two spheres. In the type IIA superstring picture, this is equivalent to the intersection of three sets of D6-branes with multiplicities m, m and n leading to $SU(m) \times SU(m) \times SU(n)$ gauge symmetry with chiral multiplets in the $(m, \bar{m}, 1) + (1, m, \bar{n}) + (\bar{m}, 1, n)$ bifundamental representations. This gauge system is represented by a quiver triangle which may be viewed as the toric geometry graph of $\mathbf{WP}^2$. On the other hand, the same physics content models can appear in type IIA superstring compactified on local elliptic fibration Calabi–Yau threefolds CY_A^3 in the presence of D6-branes wrapping 3-cycles S_i and filling the four-dimensional Minkowski spacetime [11, 12]. In terms of gauge theory, each 3-cycle S_i

is associated with a single gauge group factor and the intersection numbers (1.4) count the number of $N = 1$ chiral multiplets which transform in the bifundamental representation. We will see later that the information on the gauge system is encoded in the intersection numbers. Under mirror symmetry, the physics of D6-branes wrapping 3-cycles maps to IIB D3-branes transverse to local Calabi–Yau threefolds CY_A^3. The latter are complex cones over del Pezzo surfaces or more generally complex two-dimensional toric manifolds $\mathbf{V}^2$. In this way, the $N = 1$ four-dimensional quiver theory can be obtained from type IIB geometry using the so-called (p, q) brane webs. Indeed, the toric skeletons of these varieties are defined by the (p, q) web charges of D5-branes. They correspond to the loci of points at which some 1-cycles of the elliptic curve fibration of $\mathbf{V}^2$ shrinks to zero radius [32]. The physics content of these models can be determined explicitly from the geometry of the (p, q) webs. For more details on this method, see [14, 15]. In particular, if the vanishing 1-cycles of the elliptic fibration $C_i \equiv (p_i, q_i)$, the intersection numbers, in type IIA mirror geometry, read

$$I_{ij} = C_i \cdot C_j = p_i q_j - p_j q_i. \tag{4.1}$$

The set of the ranks of the gauge groups n_i is a null vector of this matrix, i.e

$$\sum_i (p_i q_j - p_j q_i) n_i = 0. \tag{4.2}$$

Until this level, the connection between the method of (p, q) brane webs and the Acharya–Witten model [25] is not obvious. We propose that this connection requires the introduction of the Mori vector charge Q_i^a in the description of (p, q) webs. Our solution was inspired by the following.

(1) The study of M-theory on the local geometry of Calabi–Yau threefolds having toric realization in terms of (p, q) D5-branes of type IIB superstrings [32]. In this way, the D-brane charges are associated with the vanishing cycles in the toric representation.
(2) The result of Acharya–Witten on M-theory on G_2 manifolds, where the set of ranks of gauge groups coincide with the weight vector of the $\mathbf{WP}^2$ [25].
(3) The local mirror symmetry application in type II superstrings, where the mirror constraint equations involve the toric geometry data of the original manifolds [37–40].

Besides these points, a close examination of the formulation of the (p, q) webs reveals, however, that the matrix intersection (4.1) appears in the ordinary and weighted projective spaces. Moreover, it does not carry any transparent toric geometry data distinguishing these geometries. Taking into account this observation, the connection we are after leads us to reformulate the intersection number structures by introducing the toric geometry Mori vectors Q_i^a and a set of brane charge constraint equations. To make connection with [25], we restrict ourselves to the weighted projective spaces where $\vec{Q}^1 = (w_1, w_2, w_3)$. Given a set of charges (p_i, q_i), $i = 1, 2, 3$, we propose the intersection number formula

$$\mathcal{I}_{ij} = w_i w_j (p_i q_j - p_i q_j) \tag{4.3}$$

with the following constraint equations:

$$w_1^2 p_1 + w_2^2 p_2 + w_3^2 p_3 = 0 \qquad w_1^2 q_1 + w_2^2 q_2 + w_3^2 q_3 = 0. \tag{4.4}$$

Now, the set of ranks of the gauge groups n_i should satisfy the following constraint:

$$\sum_i \mathcal{I}_{ij} n_i = 0 \tag{4.5}$$

as required by the anomaly cancellation condition [14, 15]. Using equation (4.4), it is easy to see that this condition can be satisfied in terms of the weights of $\mathbf{WP}^2$ as follows:

$$n_i = w_i n \tag{4.6}$$

and so the corresponding gauge symmetry is given by

$$G = \prod_{i=1} U(w_i n). \tag{4.7}$$

Our reformulation of the (p, q) webs has the following nice features.

(1) This formulation is quite similar to the geometric engineering of four-dimensional $N = 2$ superconformal field theories with a gauge group $G = \prod_{i=1} SU(s_i n)$ where the s_i are the usual Dynkin labels that are a null vector of affine Cartan matrices as required by the vanishing of the beta function.

(2) For $w_i = 1$, we recover the simple model with a gauge group $U(n)^3$ and matter triplication in each bifundamental [20, 21].

(3) For $n = 1$, the corresponding gauge theory is now quite similar to the interpretation of M-theory on G_2 manifolds given in [25]. In this way, the gauge group reads
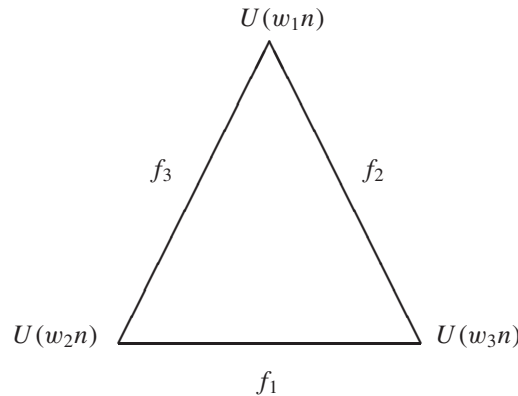
$$G = \prod_{i=1}^{3} U(w_i). \tag{4.8}$$

In the infrared limit the $U(1)$ factors decouple and one is left with the gauge symmetry

$$G = \prod_{i=1}^{3} SU(w_i). \tag{4.9}$$

Taking an appropriate choice of weights, we recover the physical model given in [25].

(4) The corresponding field model is represented by a triangle quiver diagram

Summarizing, many aspects of the physics of M-theory on such G_2 manifolds are reproduced by this formulation of type IIB superstring on complex lines on $\mathbf{V}^2$ in the presence of (p, q) brane charges, suggesting equivalence of the two descriptions. Performing local mirror symmetry, we end with local elliptic type IIA geometry with D6-branes filling four-dimensional spacetime. The corresponding gauge group and quiver diagram can be obtained using the (p, q) webs toric geometry data of $\mathbf{V}^2$.

5. Illustrating models

In this section, we will give two illustrating applications. They concern the examples studied in section 3: $\mathbf{WP}^2_{1,2,1}$ and $\mathbf{WP}^2_{1,3,2}$.

5.1. $U(n)^2 \times U(2n)$ gauge theory

Consider, first, the geometry of $\mathbf{WP}^2{}_{1,2,1}$ in M-theory compactifications. In (p, q) webs, this is equivalent to taking three stacks of branes each, wrapping the following 1-cycles

$$C_1 = (-2, 0) \qquad C_2 = (0, -1) \qquad C_3 = (2, 4). \qquad (5.1)$$

In this case, the intersection numbers read

$$\mathcal{I}_{12} = 4 \qquad \mathcal{I}_{31} = 8 \qquad \mathcal{I}_{23} = 4. \qquad (5.2)$$

For one D6-brane, this example leads to an $N = 1$ spectrum with a gauge group $U(1)^2 \times U(2)$ and bifundamental matter. This model agrees with the result of Acharya and Witten given in [25]. While for n D6-branes, the above charge configurations give an $N = 1$ spectrum with a gauge group $U(n)^2 \times U(2n)$ and bifundamental matter.

5.2. $U(n) \times U(2n) \times U(3n)$ gauge model

The geometry of $\mathbf{WP}^2{}_{1,3,2}$ is very exciting in this analysis because it may lead to the symmetry of the grand unified theory (GUT)[2]. For this example, we consider three stacks of n D6-branes each, wrapping the following 1-cycles:

$$C_1 = (4, 9) \qquad C_2 = (-1, 0) \qquad C_3 = (0, -1). \qquad (5.3)$$

In this case, the intersection numbers read

$$\mathcal{I}_{12} = 18 \qquad \mathcal{I}_{31} = 12 \qquad \mathcal{I}_{23} = 6. \qquad (5.4)$$

This yields an $N = 1$ spectrum with a gauge group $U(n) \times U(2n) \times U(3n)$ and bifundamental matter. For $n = 1$, one gets $U(1) \times U(2) \times U(3)$ as gauge symmetry.

Concluding this section, it is interesting to make a comment regarding the numbers appearing in (5.2) and (5.4), counting the number of $N = 1$ chiral multiplets f_i in the corresponding gauge systems. The latter have a remarkable feature which has a nice interpretation using the recent derivation of local mirror symmetry in two-dimensional field theory with $N = 2$ supersymmetry [39]. Indeed, in the above two examples, f_i can be written as follows.

$$f_1 = w_2 w_3 d \qquad f_2 = w_1 w_3 d \qquad f_3 = w_1 w_2 d \qquad (5.5)$$

where d is the degree of the following homogeneous LG Calabi–Yau superpotentials

$$y^2 + x^4 + z^4 + e^t xyz = 0 \qquad y^2 + x^3 + z^6 + e^t xyz = 0 \qquad (5.6)$$

mirror to type IIB $N = 2$ sigma model on the anti-canonical line bundles over $\mathbf{WP}^2{}_{1,2,1}$ and $\mathbf{WP}^2{}_{1,2,3}$, respectively.

6. Conclusion

In this paper, we have studied $N = 1$ supersymmetric gauge theories embedded in M-theory on local seven-dimensional manifolds with G_2 holonomy group. We have engineered the $N = 1$ quiver models from G_2 manifolds, as $U(1)$ quotients of eight-dimensional toric HK manifolds. The corresponding quiver models have been obtained using a reformulation of the method of (p, q) webs. Our main results may be summarized as follows.

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[2] This is not a surprise since it was shown in [51] that one gets the GUT gauge symmetry from M-theory on G_2 manifolds with A_4 singularity.

(i) Using two-dimensional $N = 4$ sigma models with $U(1)^r$ gauge symmetry and $r + 2$ hypermultiplets, we have constructed a special kind of G_2 manifold. The latter are $U(1)$ quotients of eight-dimensional toric (HK) manifolds, $X_7 = \frac{X_8}{U(1)}$. We have shown that these seven-dimensional manifolds, in general, are given by real cones on $\mathbf{S}^2$ bundles over complex two-dimensional toric varieties $\mathbf{V}^2 = \mathbf{C}^{r+2}/\mathbf{C}^{*r}$. Explicit models have been given in terms of $N = 2$ sigma model realizations of $\mathbf{V}^2$.

(ii) We have discussed the link between the physics content of M-theory on such G_2 manifolds and the method of (p, q) webs. We have reconsidered and reformulated the method of (p, q) webs using the toric geometry Mori vectors of $\mathbf{V}^2$ and brane charge constraint equations. For the weighted projective space $\mathbf{WP}^2{}_{w_1,w_2,w_3}$, we have found that the corresponding field model has $G = U(w_1 n) \times U(w_2 n) \times U(w_3 n)$ as the gauge symmetry group. This is required by the anomaly cancellation condition.

Acknowledgments

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